F. anal. 3.2.2025

Mail: Øv.-grp.
Frist lf
Ref.-g'p.

Chernoff bevis
Hölder bevis
BV bevis
Regularity vs decay
Appl. PDEs conv.

- Info, slides

A. Series and bases in $\mathcal{R}^2$ (or L^2)

$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f \times \bar{g}$, $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f|^2$

ONS = orthonormal seq'nce (w.r.t. $(\cdot,\cdot)$ in L^2)

Lem. 1: $f \in \mathcal{R}^2$ (or L^2), $\{\varphi_n\}_{n\in\mathbb{Z}} \subset \mathcal{R}^2$ (or L^2) ONS.

Then

$$(1) \quad \|f - \underbrace{\pi_N(f)}_{\sum_{|n|\le N}(f,\varphi_n)\varphi_n}\| \to 0 \qquad (L^2\text{-conv.})$$

$$\Updownarrow$$

$$(2) \quad \|f\|^2 = \sum_{n\in\mathbb{Z}} |(f,\varphi_n)|^2 \qquad \text{(Plancherel)}$$

Pf: By Lem 2 last time: $\|f\|^2 = \|f - \pi_N(f)\|^2 + \sum_{|n|\le N} |(f,\varphi_n)|^2$ ✓ Send $N\to\infty$ □

Rem. 1:

(a) (1) $\Leftrightarrow \sum_{n\in\mathbb{Z}}(f,\varphi_n)\varphi_n$ conv. in $\|\cdot\|$, and $= f$:

$\Leftrightarrow f = \sum_{n\in\mathbb{Z}}(f,\varphi_n)\varphi_n$ in L^2 $(\|f - \Sigma\| = 0)$

(b) If (2) holds, then $\|f\| = 0 \Leftrightarrow (f,\varphi_n) = 0 \ \forall n\in\mathbb{Z}$

Def. 1:

(a) $\{\varphi_n\}_{n\in\mathbb{Z}}$ <u>orthonormal basis</u> (= ONB) if ONS and (1) holds

(b) $\sum_{n\in\mathbb{Z}}(f,\varphi_n)\varphi_n$ / (f,φ_n): <u>generalized Fourier series</u> / <u>coeff.</u> when $\{\varphi_n\}_n$ ONB

B. $\|\cdot\|$-conv. for F.-series (L^2-theory)

$\varphi_n = e_n = e^{inx}$, $\hat{f}(n) = (f, e_n)$; $S_N(f) = \pi_N(f)$

$\{e_n\}_{n\in\mathbb{Z}}$ ONS in $\mathcal{R}^2(0,2\pi)$ ($L^2(0,2\pi)$) (last time)

Thm. 1: If $f \in \mathcal{R}^2$ (or L^2), then

$$\|f - S_N(f)\| \underset{N\to\infty}{\longrightarrow} 0.$$

Cor. 1:

(a) $f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$ in L^2 ($\|f - \Sigma\| = 0$)

(b) $\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$ (Plancherel)

(c) $\{e_n\}$ ONB in $\mathcal{R}^2$ (and L^2)

Pf.: Lem. 1, Rem. 1, Def. 1 (a) $\square$

Rem. 2 Ex. 1 p. 3

Pf. Thm 1:

1) (Prev.) $\exists \{f_m\}_m \subset C([0,2\pi])$ s.t. $\|f - f_m\|^2 = \frac{1}{\sqrt{2\pi}}\|f - f_m\|_2 \underset{m\to\infty}{\longrightarrow} 0$

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2) $\|S_N(f) - S_N(f_m)\|^2 = \|S_N(f - f_m)\|^2$

$\overset{\{e_n\}\text{ ONS}}{=} \sum_{|n|\le N} |(f - f_m, e_n)|^2 \overset{\text{Bessel}}{\le} \|f - f_m\|^2$

3) f_m cont. $\Rightarrow \exists$ trig. poly. $\{P_{mn}\}$ of order n s.t. $\|f_m - P_{mn}\|_\infty \underset{n\to\infty}{\longrightarrow} 0$

(e.g. $P_{mn} = \sigma_n(f_m)$, Cesàro sum)

4) By best approx. (last time):

(continued on p. 4)

Rem. 2:

(a) $\sin nx = \frac{1}{2i}(e_n - e_{-n})$, $\cos nx = \cdots$

$\overset{\text{Cor 1 (c)}}{\Longrightarrow}$ $\{\cos nx, \sin nx\}_{n=0}^{\infty}$ ONB in $\mathcal{R}^2(0,2\pi)$

$(L^2(0,2\pi))$

(b) Cor 1 (c) + odd/even extensions

$\Longrightarrow$ $\{\sin nx\}_{n=1}^{\infty}$ and $\{\cos nx\}_{n=0}^{\infty}$ both ONB in $\mathcal{R}^2(0,\pi)$

$(L^2(0,\pi))$

[$\{e_{\frac{n}{2}}\}_{n\in\mathbb{Z}}$ also ONB in $L^2(0,\pi)$]

(c) Plancherel: Find sums of series (Mat 4)

Ex. 1:

$f(x) = x$, $x \in [-\pi, \pi]$

$S(f)(x) \overset{\text{Prev.}}{=} \sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{inx}$ (F.-series)

If $S_N(f) \to f$ in $\|\cdot\|$, then

Cor. 1 (a) $\Rightarrow$ $\|f - S(f)\| = 0$, $\overset{x=f}{f = S(f)}$ in L^2

Plancherel:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n\neq 0} \frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\| $$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi} 2\left[\tfrac{1}{3}x^3\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \underline{\underline{\frac{\pi^2}{6}}}$$

Back to p. 2

$$\|f_m - S_N(f_m)\| \overset{\text{chk.}}{\leq} \|f_m - P_{mN}\| \leq \|f_m - P_{mN}\|_\infty \xrightarrow[N\to\infty]{3)} 0$$

5) $\|f - S_N(f)\| \leq \|f - f_m\| + \|f_m - S_N(f_m)\| + \|S_N(f_m) - S_N(f)\|$

$$\overset{2)}{\leq} 2\|f - f_m\| + \|f_m - S_N(f_m)\|$$

First take m large so $2\|f - f_m\| < \frac{\varepsilon}{2}$ (by 1)), then

$N = N(m)$ large s.t. $\|f_m - S_N(f_m)\| < \frac{\varepsilon}{2}$ (by 4)) $\square$

Thm. 2: (Parseval 2)

$f, g \in \mathcal{R}^2(0, 2\pi)$ (or $L^2(0, 2\pi)$), then

$$(f, g) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \cdot \overline{\hat{g}(n)} \quad (= (\hat{f}, \hat{g})_{\ell^2})$$

"Pf.:" From Plancherel 1 since (chk)

(*) $\langle x, y\rangle = \frac{1}{4}\left(|||x+y|||^2 - |||x-y|||^2 + i|||x+iy|||^2 - |||x-iy|||^2\right)$

$\mathbb{C}$-valued i.p. induced norm $\square$

(*) in L^2 + Plancherel 1 + (*) in ℓ^2 $\square$

Rem. 3: Let $\mathcal{F}_\mathbb{Z} : \mathcal{R}^2$ (or L^2) $\to \ell^2(\mathbb{Z})$ be def. by

$$\mathcal{F}_\mathbb{Z}(f)(n) = \hat{f}(n), \; n \in \mathbb{Z}$$

(a) $\mathcal{F}_\mathbb{Z}$ lin., $\|\mathcal{F}_\mathbb{Z}(f)\|_{\ell^2} \overset{\text{Plancherel}}{=} \|f\|$

$\Rightarrow \mathcal{F}_\mathbb{Z}$ bnd. lin. op., $\|\mathcal{F}_\mathbb{Z}\| = 1$

(b) $\mathcal{F}_\mathbb{Z} : L^2(0, 2\pi) \to \ell^2(\mathbb{Z})$, 1-1, onto (isomorphism)

(c) $L^2(0,2\pi) \overset{\text{isomorphic}}{\simeq} \ell^2(\mathbb{Z})$ (by b))

$f \in L^2$ correspond uniquely to $\hat{f} \in \ell^2$

"As Hilbert sp's $L^2(0,2\pi)$ and $\ell^2(\mathbb{Z})$ are the same"

C. Local/pt. wise conv. of F.-series

One-sided limits and derivatives:

$$f(x_0\pm) = \lim_{h\to 0+} f(x_0 \pm h)$$

$$D^+f(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} \quad \text{(right deriv.)}$$

$$D^-f(x_0) = \lim_{h\to 0^+} \frac{f(x_0)-f(x_0-h)}{h} \quad \text{(left deriv.)}$$

General result:

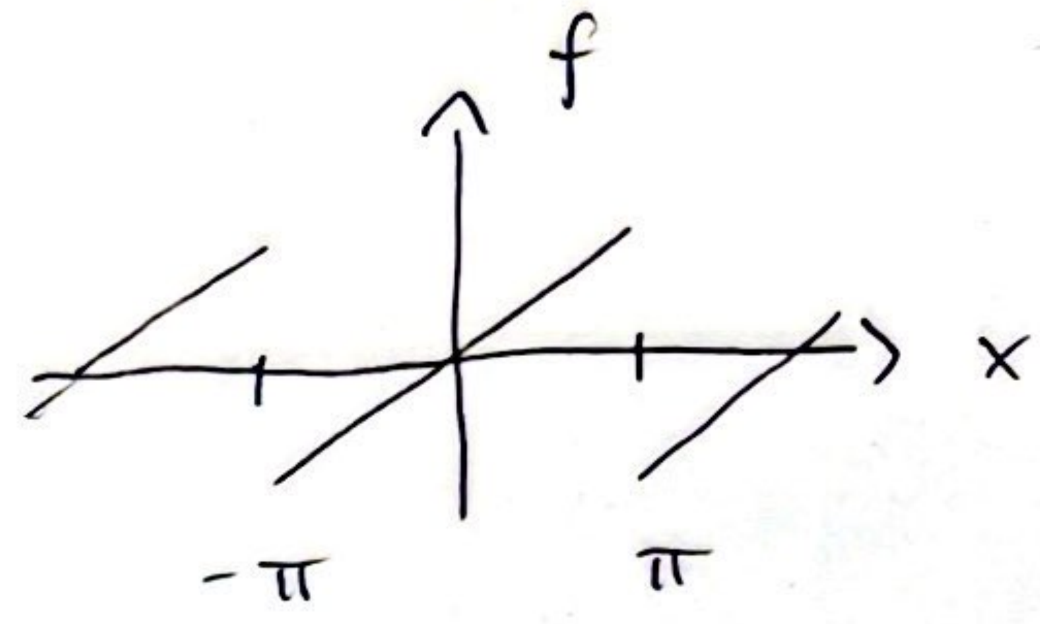
Thm. 3: f R.-int. (L^1) and $f(x_0\pm)$, $D^\pm f(x_0)$ exist.

Then $S_N(f)(x_0) \longrightarrow \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right)$.

Rem. 4: f cont. at $x_0 \Longrightarrow \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right) = f(x_0)$

Ex. 1 (cont.)

$f(x) = x, \quad x \in (-\pi, \pi]$



$f(x^\pm) = f(x) = x, \quad x \in (-\pi, \pi) \quad$ (f cont. here)

$f(\pi^-) = \pi$

$f(\pi^+) = f(\pi^+ - 2\pi) = f((-\pi)^+) = -\pi$

$D^\pm f(x) = f'(x) = 1, \quad x \in (-\pi, \pi) \quad$ (f' ex. here)

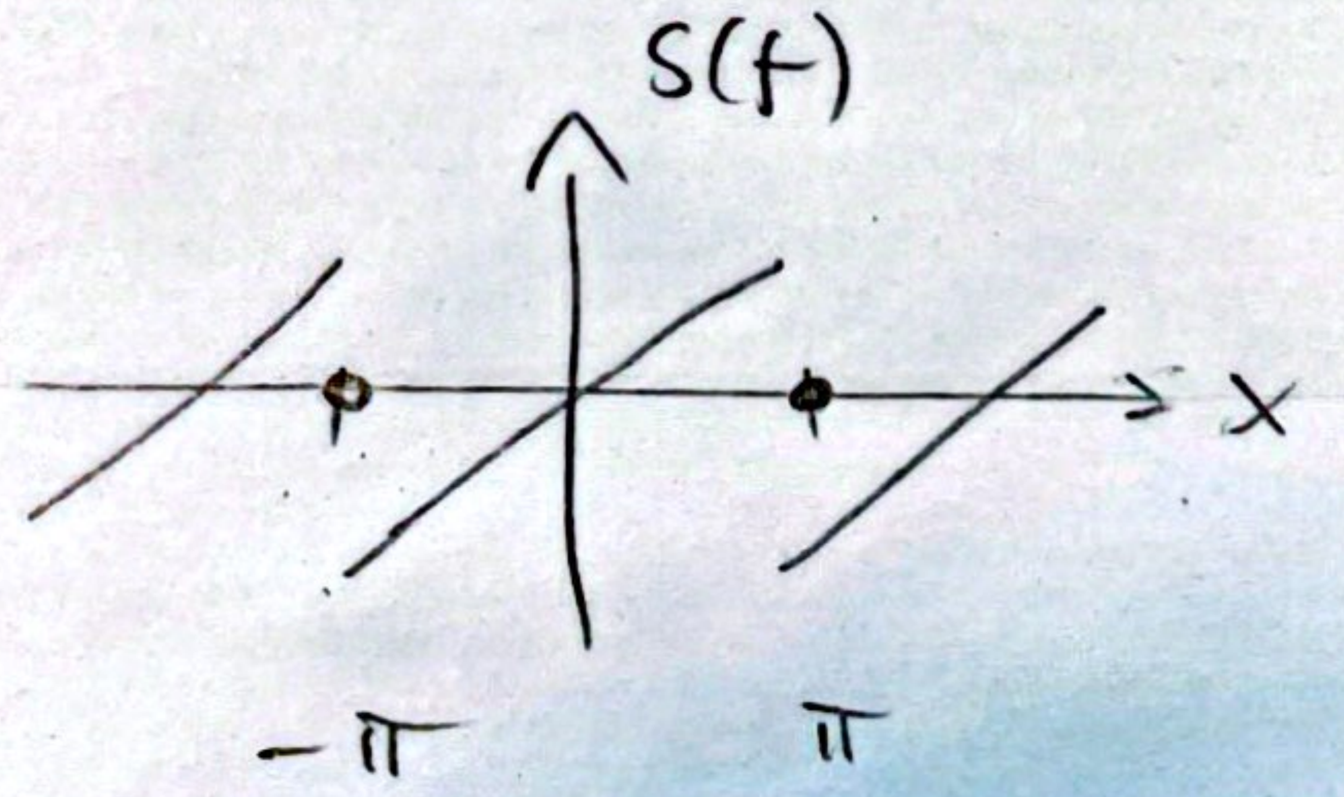
$D^- f(\pi) = \lim\limits_{x \to \pi^-} f'(x) = 1$

$D^+ f(\pi) = D^+ f(-\pi) = \lim\limits_{x \to (-\pi)^+} f'(x) = 1$

By Thm. 1:

$S(f)(x) = \lim\limits_{N \to \infty} S_N(f)(x) = \frac{1}{2}(f(x) + f(x)) = f(x) = x,$
$x \in (-\pi, \pi)$

$S(f)(\pi) = \frac{1}{2}(f(\pi^-) + f(\pi^+)) = \frac{1}{2}(0.0) = 0$



F. anal. 6.2.2025

- Slides, info

A. Local/pt. wise conv. for F.-series

Thm. 1: f R.-int. (L^1) and $f(x_0\pm)$, $D^\pm f(x_0)$ exist.

Then $S_N(f)(x_0) \longrightarrow \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right)$

Rem. 1: The pf. uses R.-L.-lem. on

$$g(y) := \begin{cases} \dfrac{f(x_0-y) - f(x_0^+)}{e^{iy}-1}, & y \in (-\pi, 0) \\[2ex] \dfrac{f(x_0-y) - f(x_0^-)}{e^{iy}-1}, & y \in (0, \pi) \end{cases}$$

R.-L.-lem. holds for R-int. f'ns (and L^2),

but also in L^1 (approx. by cont. f'ns) $\to$ Ex??

<u>Lem 1:</u> f p.w. cont./R.int. (/L^1) $\overset{\text{on } S}{}$ and $\overset{f(x_0^\pm),}{D^\pm f(x_0)}$ exists

$\Rightarrow$ g p.w. cont./R.int. (/L^1) on S

<u>Pf.:</u>

$$(*)\quad \lim_{y\to 0^\pm} g(y) = \lim_{y\to 0^\pm} \frac{y}{e^{iy}-1} \cdot \lim_{y\to 0^\pm} \frac{f(x_0-y)-f(x_0^\mp)}{y} = \frac{1}{i}(\mp D^\mp) f(x_0)$$

(l'Hôpital)

f p.w. cont. $\Rightarrow$ g p.w. cont. for $y\neq 0$ (where $e^{iy}-1\neq 0$)

and by $(*)$ for $y=0$

f R.int. $\Rightarrow$ $\frac{1}{e^{iy}-1}$, f, g R.int. (and bnd.) for $|y|>\delta$ $\forall \delta\in(0,\pi)$

$(*)$ $\Rightarrow$ g bnd for $|y|<\delta$ (δ small) [conv. $\Rightarrow$ bnd]

bnd + R-int. for $|y|>\delta$ $\forall\delta>0$ $\overset{\text{Prop.1.4 in App. SS}}{\Rightarrow}$ R-int. in $(-\pi,\pi)$ $\square$

<u>Lem.2:</u>

(a) $D_N(x) = \left(\sum_{n=-N}^{-1} + \sum_{n=0}^{N}\right) e^{inx} = \dfrac{e^{i(N+1)x} - e^{-iNx}}{e^{ix}-1} \overset{\text{prev.}}{=} \dfrac{\sin((N+\frac{1}{2})x)}{\sin\frac{x}{2}}$

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N \overset{\text{prev.}}{=} 1 \overset{D_N \text{ even}}{\Longrightarrow} \frac{1}{2\pi}\int_0^{\pi} D_N = \frac{1}{2\pi}\int_{-\pi}^{0} D_N = \frac{1}{2}$

<u>Pf. of Thm. 1:</u>

$$\tfrac{1}{2}(f(x_0^-)+f(x_0^+)) = f(x_0^+)\frac{1}{2\pi}\int_{-\pi}^{0} D_N + f(x_0^-)\frac{1}{2\pi}\int_0^{\pi} D_N$$

$$D_N * f(x_0) \overset{\text{def}*}{=} S_N(f)(x_0) - \tfrac{1}{2}\left(f(x_0^-) + f(x_0^+)\right)$$

$$\overset{\text{Lem. 2 b)}}{=} \frac{1}{2\pi}\int_{-\pi}^{0}[f(x_0-y) - f(x_0^+)]\,D_N(y)\,dy + \frac{1}{2\pi}\int_{0}^{\pi}[f(x_0-y) - f(x_0^-)]\,D_N(y)\,dy$$

$$\overset{\text{Lem 2a)}}{\underset{\text{Def. of } g}{=}} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\left(e^{i(N+1)y} - e^{-iNy}\right)dy$$

$$\underset{\text{def. } \hat{g}}{=} \underbrace{\hat{g}(-(N+1))}_{\downarrow\, 0} - \underbrace{\hat{g}(N)}_{\downarrow\, 0} \xrightarrow[N\to\infty]{\text{Lem. 1 + R.-L.-lem}} 0$$

$\square$

B. Global/uniform conv. of F-series

Recall:

Lem. 3: $f \in C^1(\mathbb{S}) \Rightarrow \widehat{f'}(n) = in\hat{f}(n)$

Hence:

(**) $f \in C^k(\mathbb{S}), k \in \mathbb{N} \Rightarrow \widehat{f^{(k)}}(n) = (in)^k \hat{f}(n)$

Thm. 2: $f \in C^k(\mathbb{S}), k \in \mathbb{N}$. Then

$$\|f - S_N(f)\|_\infty \leq \frac{\|f^{(k)}\|_\infty}{\sqrt{k-\frac{1}{2}}} N^{-k+\frac{1}{2}} \quad \left(\xrightarrow[N\to\infty]{} 0\right).$$

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Cor. 1: $f \in C^1(\mathbb{S}) \Rightarrow \|f - S_N(f)\|_\infty \xrightarrow[N\to\infty]{} 0$

[Thm. 2 with $k=1$]

In particular if $f \in C^1(\mathbb{S})$, then

(a) $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$ conv. unif. (and abs.) in $\mathbb{S}$

(b) $\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} = f(x) \ \forall x \in \$$

Rem. 2:

(a) From pf.: $\sum |\hat{f}(n)| \le c\|f'\|_\infty \underset{\text{Thm. 2}}{\Rightarrow}$ F.-series abs. conv.

(b) More regular $f \Rightarrow S_N(f)$ conv. faster

(c) General result: f Hölder cont., $|f(x)-f(y)| \le K|x-y|^\alpha$ for $\alpha \in (\frac{1}{2},1] \Rightarrow S_N(f) \xrightarrow[\text{abs.}]{\text{unif.}} f$

Cor. 2: $f \in C^k(\$)$, $k \in \mathbb{N}$, $m = 1,2,\dots,k-1$.

(a) $S_N(f^{(m)}) \xrightarrow[\text{abs.}]{\text{unif.}} f^{(m)}$ on $\$$

(b) $f^{(m)}(x) \overset{(a)}{=} \sum_{n\in\mathbb{Z}} \widehat{f^{(m)}}(n) e^{inx} \underset{\text{Lem. 3}}{=} \sum_{n\in\mathbb{Z}} (in)^m \hat{f}(n) e^{inx}$

Pf.: $f^{(m)} \in C^1(\$)$ + Cor. 1 $\Rightarrow$ (a)

(a) + Lem. 3 $\Rightarrow$ (b) $\square$

Cor. 2 b): Term-wise differentiation of F.-series!

Pf. of Thm. 2:

1) Since $f \in C^1(\$)$, by Thm. 1,

$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}, \ \forall x \in \$$

2) $|f(x) - S_N(f)(x)| = |\sum_{n\in\mathbb{Z}} \dots - \sum_{|n|\le N} \dots| = |\sum_{|n|>N} \dots|$

$\overset{\Delta\text{-ineq.}}{\le} \sum_{|n|>N} |\hat{f}(n)| \cdot \underbrace{|e^{inx}|}_{=1} \overset{\text{Lem. 3}}{=} \sum_{|n|>N} \frac{1}{|n|^k} \underbrace{|(in)^k \hat{f}(n)|}_{\underset{\text{Lem. 3}}{=} |\widehat{f^{(k)}}(n)|}$

$\overset{\text{C-S in } \ell^2(\mathbb{Z})}{\le} \underbrace{\Big(\sum_{|n|>N} \frac{1}{|n|^{2k}}\Big)^{\frac{1}{2}}}_{\le 2\int_N^\infty \frac{1}{x^{2k}}dx = \frac{2}{2k-1} N^{-2k+1}} \underbrace{\Big(\sum_{|n|>N} |\widehat{f^{(k)}}(n)|^2\Big)^{\frac{1}{2}}}_{\underset{\text{Bessel}}{\le} \|f^{(k)}\|^2 \le \|f^{(k)}\|_\infty^2}$

Hence

$$\|f - S_N(f)\|_\infty \le \frac{1}{\sqrt{k-\frac{1}{2}}} N^{-k+\frac{1}{2}} \|f^{(k)}\|_\infty \qquad \square$$

C. Regularity of $f(x)$ vs. decay of $\hat{f}(n)$

α-Hölder $\Rightarrow O(\frac{1}{n^\alpha})$ Øving

$C^{k+\alpha} \Rightarrow O(\frac{1}{n^{k+\alpha}})$

$\widehat{f^{(k)}}(n) = O(\frac{1}{n^\alpha})$

Regularity $\Rightarrow$ decay:

Lem. 4:

(a) $f \in C^k(\$),\ k \ge 1 \Rightarrow \hat{n}^k |\hat{f}(n)| \underset{n\to\infty}{\longrightarrow} 0$

(b) $\alpha \in (0,1]$, f α-Hölder $\overset{\text{cont.}}{\vee}$: $|f(x) - f(y)| \le K|x-y|^\alpha$, $x, y \in \$$ (α-Hölder)

$\Rightarrow |\hat{f}(n)| \le \frac{C}{|n|^\alpha} \quad (n \neq 0)$

(c) $f^{(k)}$ α-Hölder $\Rightarrow |\hat{f}(n)| \le \frac{C}{|n|^{k+\alpha}} \quad (n \neq 0)$

Pf.: (a) $\sum |\widehat{f^{(k)}}(n)|^2 \overset{\text{Bessel}}{\le} \|f^{(k)}\|^2 \le \|f^{(k)}\|_\infty^2 < \infty$

$\Rightarrow \underbrace{(n^k |\hat{f}(n)|)^2 = |\widehat{f^{(k)}}(n)|^2}_{\text{Lem 3}} \underset{\text{div. test}}{\longrightarrow} 0$

(b) Exercise 4

(c) $|\hat{f}(n)| = \left| \frac{(in)^k \hat{f}(n)}{(in)^k} \right| = \frac{1}{|n|^k} |\widehat{f^{(k)}}(n)| \overset{(b)}{\le} \frac{C}{|n|^{k+\alpha}} \qquad \square$

Decay $\Rightarrow$ regularity:

Lem 5: If f R. int. (or L^1), $k \in \mathbb{N}$, $\beta > 1$, and

$$|\hat{f}(n)| \le \frac{C}{|n|^{k+\beta}} \quad \forall n \in \mathbb{Z}.$$

Then $f \in C^k(\$)$.

"Pf.:"

$$j \le k: \sum_{n \ne 0} |(in)^j \hat{f}(n)| \le C \sum_{n \ne 0} |n|^{-(k-j+\beta)} \overset{j \le k}{\le} C \sum_{n \ne 0} |n|^{-\beta} \overset{\beta > 1}{<} \infty$$

$$\underset{\text{M-test}}{\overset{\text{Weierstrass}}{\Longrightarrow}} \sum_{n \in \mathbb{Z}} (in)^j \hat{f}(n) \xrightarrow[\text{abs.}]{\text{unif.}} g_j \text{, and } g_j \text{ cont. (unif. limit of cont.)}$$

Can show: $g_0 = f$, $g_j = f^{(j)}$ for $j = 1, \dots, k$. $\square$

$\left(\sum \hat{f}(n) e^{inx} \to f \text{ in } L^2/L^2 \text{ and } \overset{\text{unif}}{\to} g_0 \Rightarrow f = g_0 \right)$ (dominated conv.)

12:02

If time:

Ex. 1:

g bnd.

$$u(x,t) = \sum_{n \in \mathbb{Z}} \underbrace{e^{-n^2 t} \hat{g}(n)}_{\hat{u}(n,t)} e^{inx} \quad \text{(heat eq'n)}$$

$$\Rightarrow |\hat{u}(n,t)| \le e^{-n^2 t} \|g\|_\infty = \underbrace{\left(\frac{n^k}{e^{-n^2 t}} \right)}_{\substack{\longrightarrow 0 \ \forall k \text{ if } t>0, \\ n \to \infty}} \|g\|_\infty \frac{1}{n^k} \overset{t>0}{\le} C_{t,k} \frac{1}{n^k} \quad \forall k \in \mathbb{N}$$

so bnd.

$\Rightarrow u(\cdot, t) \in C^\infty(S)$ for $t > 0$.